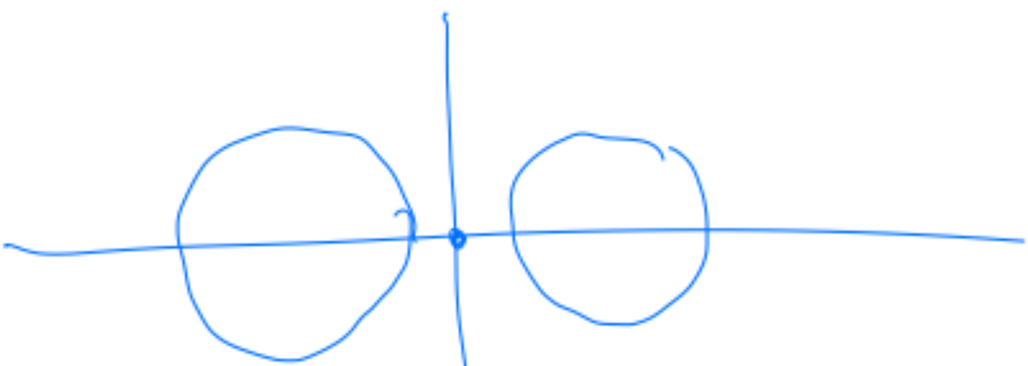
Finding the Volume of a Solid Torus

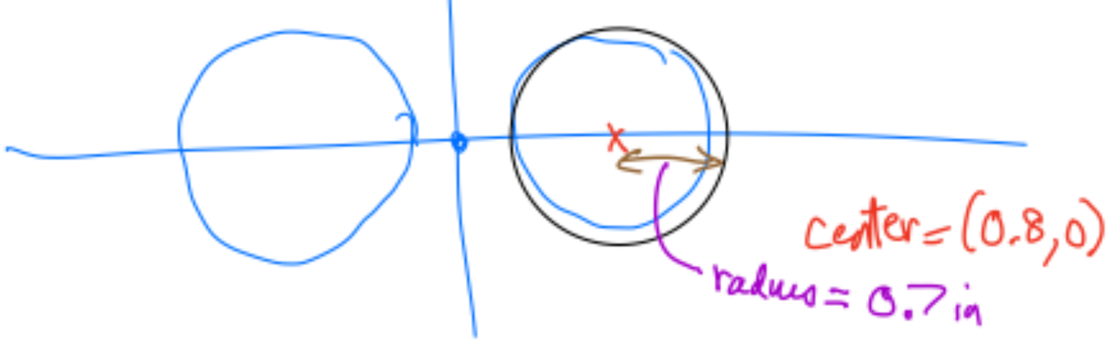
(in many ways)

Everyone needs
1. bagel
2. knife
3. graph paper. (0.1 in x 0.1 in resolution)

Step 1: Graph One cross-section.



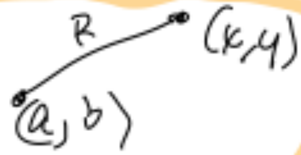
Step 2: Approximate one of the pieces by a circle with fixed center & radius:



Step 3: Write equation of circle:

$$(x-0.8)^2+y^2=(0.7)^2$$

eqn of circle

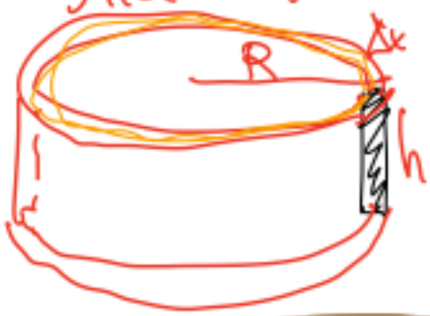


$$(x-a)^2+(y-b)^2=R^2$$

Step 5: We will calculate the volume of the perfect circle revolved around the y-axis in 3 different ways:

(A) Vertical slices in circle

Slice after revolving:



Cylindrical shell

If we cut → rectangle

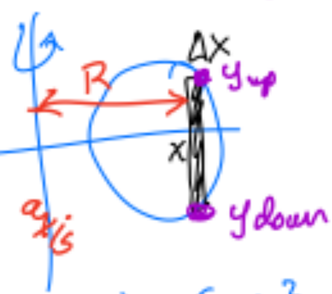
Volume of slice $=2\pi R h \Delta x$

$$\text{Total Volume} = \int (\text{Volume of slice})$$

$$= \int 2\pi R h \, dx$$

$$(x-0.8)^2 + y^2 = (0.7)^2$$

vertical dist = difference of y values.



$$h = y_{up} - y_{down}$$

Need in terms of x

$$\Rightarrow y^2 = (0.7)^2 - (x-0.8)^2$$

$$y = \pm\sqrt{(0.7)^2 - (x-0.8)^2}$$

$$y_{up} = +\sqrt{(0.7)^2 - (x-0.8)^2}$$

$$y_{down} = -\sqrt{(0.7)^2 - (x-0.8)^2}$$

$$\Rightarrow h = y_{up} - y_{down} = 2\sqrt{(0.7)^2 - (x-0.8)^2}$$

$$R = x.$$

$$\int_{0.1}^{1.5} 2\pi \overbrace{x}^{R} \overbrace{2\sqrt{(0.7)^2 - (x-0.8)^2}}^{h} \, dx$$

center = (0.8, 0)
radius = 0.7

$$\text{Total Volume} = \int_{0.1}^{1.5} 2\pi x \, 2\sqrt{(0.7)^2 - (x-0.8)^2} \, dx$$

Type some Sage code below and press Evaluate.

```
f(x)=2*pi*x*2*sqrt(0.7^2-(x-0.8)^2)
ii=integral(f(x),(x,0.1,1.5))
show(ii.n())
```

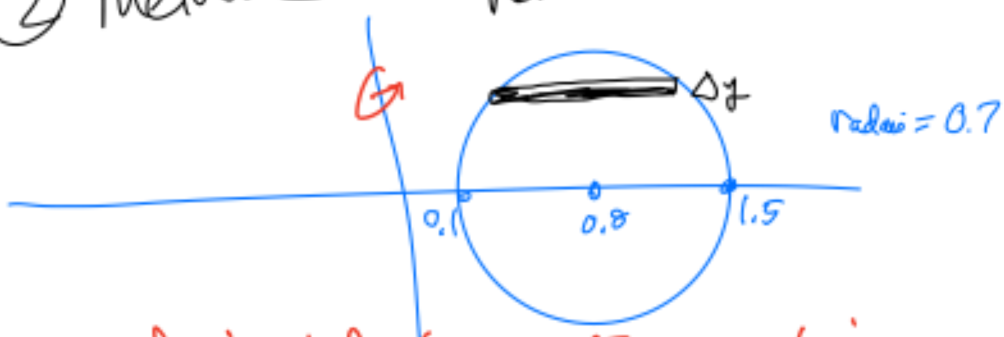
Evaluate

$7.73776979855003 - 7.45349670390816 \times 10^{-9} i$

$\boxed{\text{Volume} \approx 7.74 \text{ in}^3}$

About

2) Method 2 - horizontal slices



horizontal slice after revolving

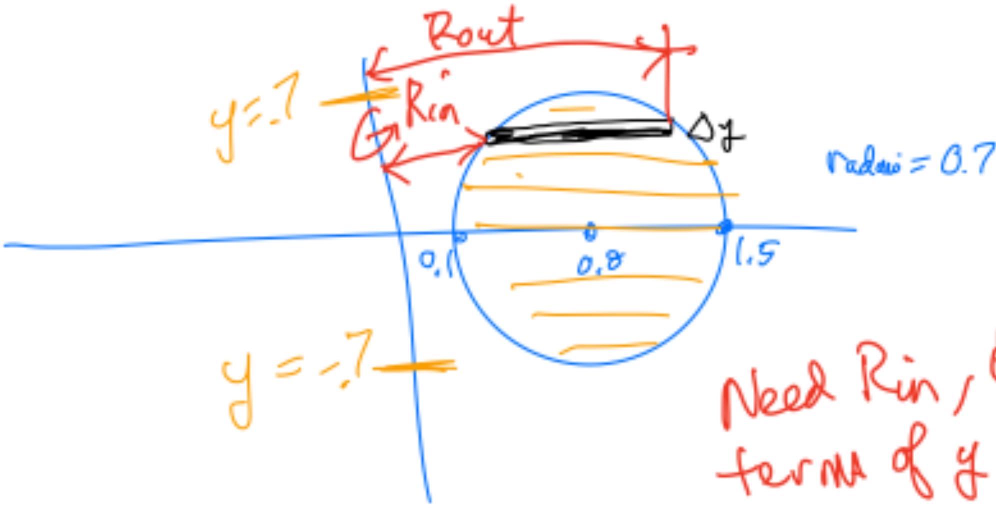
Washer shape

Thickness $= \Delta y$

$$\left(\begin{array}{c}\text{Volume of} \\ \text{slice}\end{array}\right) = \left(\pi R_{out}^2 - \pi R_{in}^2\right) \Delta y$$

$$\text{Total Volume} = \int (\text{Volume of slice})$$

$$= \int \left(\pi R_{out}^2 - \pi R_{in}^2\right) dy$$



Need Rin, Rout in terms of y. (They are horizontal distances = differences of x-values)

$$(x-0.8)^2 + y^2 = (0.7)^2$$

Need to solve for x:

$$(x-0.8)^2 = (0.7)^2 - y^2$$

$$(x-0.8) = \pm\sqrt{0.49 - y^2}$$

$$x = 0.8 \pm \sqrt{0.49 - y^2}$$

$$R_{out} = x_{big} - 0 = 0.8 + \sqrt{0.49 - y^2}$$

$$R_{in} = x_{small} - 0 = 0.8 - \sqrt{0.49 - y^2}$$

$$\text{Total Volume} = \int \left(\pi R_{out}^2 - \pi R_{in}^2\right) dy$$

$$= \int_{-0.7}^{0.7} \left[\pi\left(0.8+\sqrt{0.49-y^2}\right)^2 - \pi\left(0.8-\sqrt{0.49-y^2}\right)^2\right] dy$$

Type some Sage code below and press Evaluate.

```
f(y)=pi*(0.8+sqrt(0.49-y^2))^2-pi*(0.8-sqrt(0.49-y^2))^2
ii=integral(f(y),(y,-0.7,0.7))
show(ii.n())
```

Evaluate

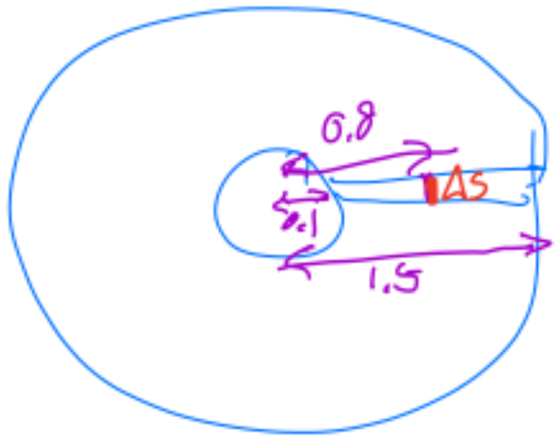
7.73776990288504

$$\boxed{\text{Volume} \doteq 7.74 \text{ in}^3}$$

Method 3

slice into thickened circles

bagel

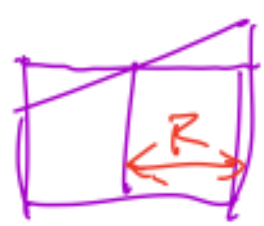


slice

side view of slice

Δs thickness in middle

side view

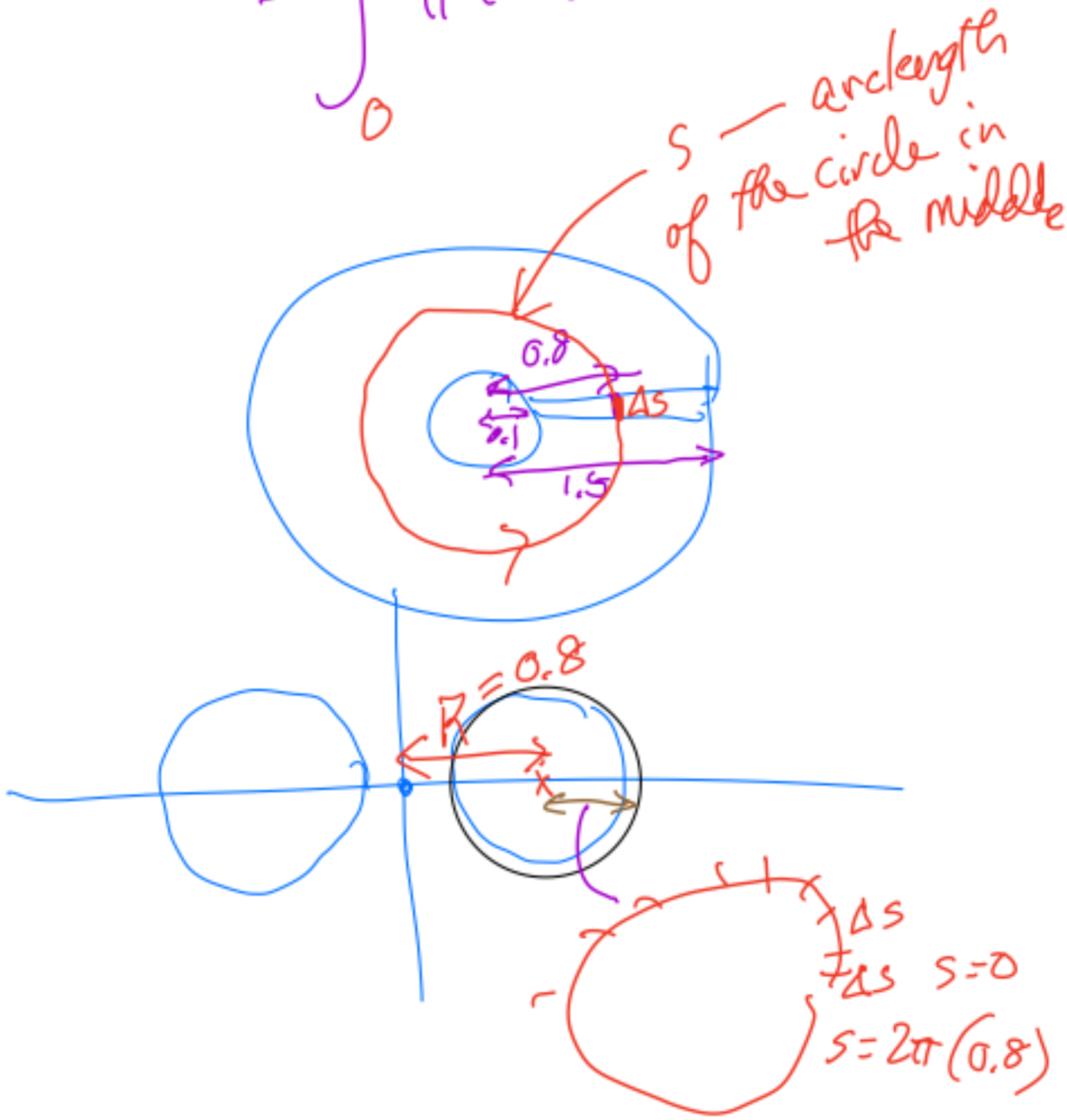


$$\text{volume of slice} = (\pi R^2)\Delta s$$

$$R = 0.7\text{ in}$$

Total Volume $= \int (\text{Volume of slice})$

$$= \int_0^{2\pi(.8)} \pi (0.7)^2 \, ds$$



$$\text{Total} = \int_{s=0}^{2\pi(0.8)} \pi(0.49)\,ds$$

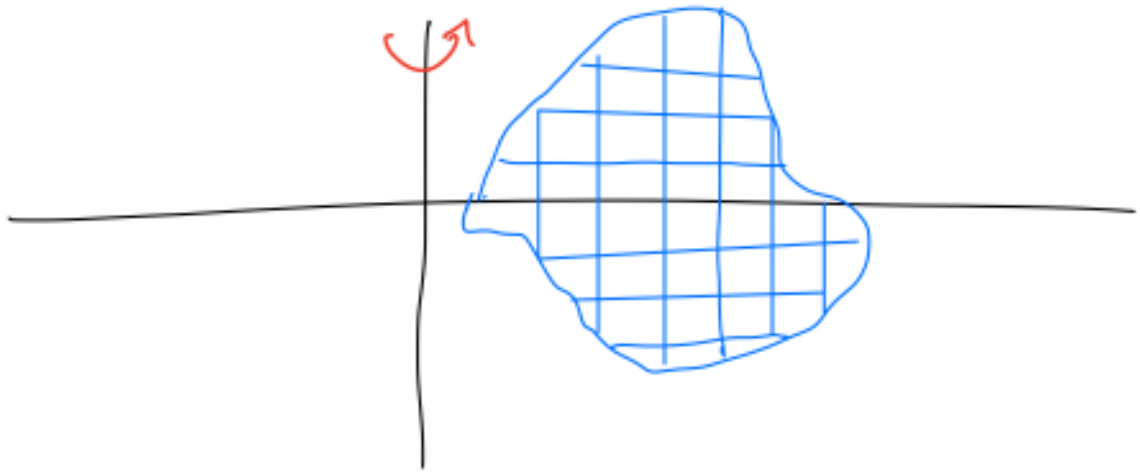
$$= \pi(0.49)\,s \Big|_0^{2\pi(0.8)}$$

$$= \pi(0.49)(2\pi(0.8)) - 0$$

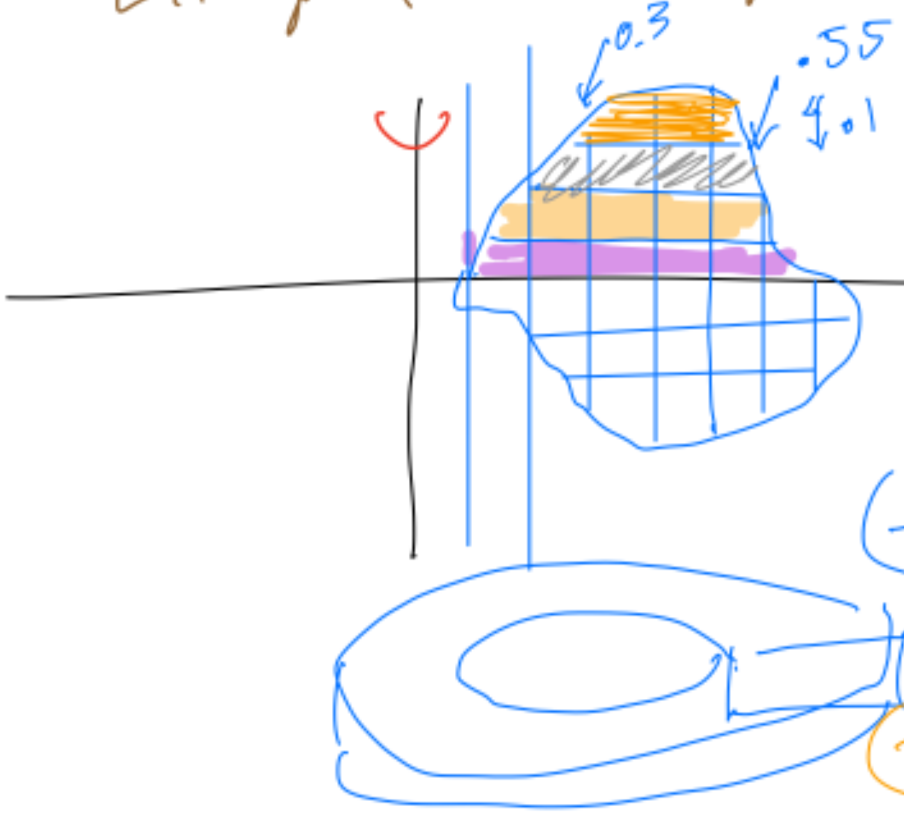
$$= 2\pi^2(0.49)(0.8)$$

$$= \boxed{7.74 \text{ in}^3}$$

Step 6: Estimating an irregular volume rotated.



Example (Washer shape → horiz slices)



$(\pi R_{out}^2 - \pi R_{in}^2)\Delta y$

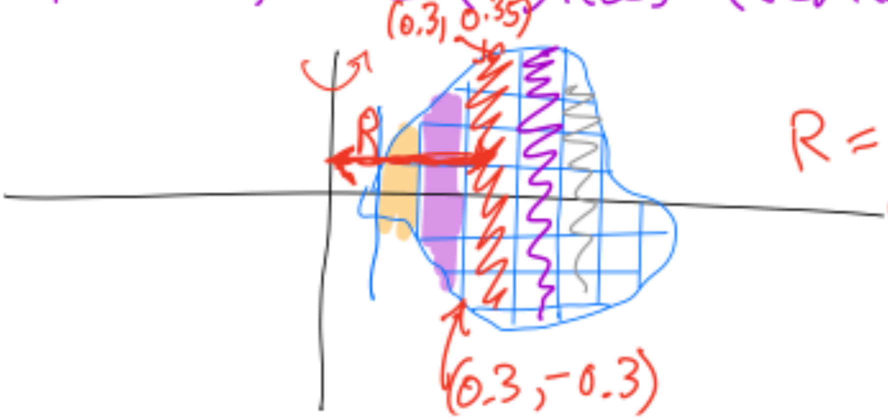
Orange slice

$(\pi(.55^2) - \pi(.3)^2).1$

Volume approx $= \sum \left(\pi R_{out}^2 + \pi R_{in}^2\right)\Delta y$

add these slice volumes.

Alternately: Shell slices (vertical slices)



$R = 0.35$

$h = 0.35 - (-0.3)$

$= 0.65$

$\Delta x = 0.1$

red slice

$$(2\pi R h)\Delta x$$

$$= 2\pi(0.35)(0.65)(.1)$$

Total Volume: add up!

Total Vol =

$$(2\pi)(.1)\left[(\overset{R}{\ })(\overset{h}{\ }) + (\overset{R}{\ })(\overset{h}{\ }) + \cdots\right]$$